Published by Institute of Physics Publishing for SISSA
Received: *March 9, 2006*
Accepted: *March 20, 2006*
Published: *March 30, 2006*

Spinning Hopf solitons on $S^3 \times \mathbb{R}$

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Abstract: We consider a field theory with target space being the two dimensional sphere S^2 and defined on the space-time $S^3 \times \mathbb{R}$. The Lagrangean is the square of the pull-back of the area form on S^2. It is invariant under the conformal group $SO(4,2)$ and the infinite dimensional group of area preserving diffeomorphisms of S^2. We construct an infinite number of exact soliton solutions with non-trivial Hopf topological charges. The solutions spin with a frequency which is bounded above by a quantity proportional to the inverse of the radius of S^3. The construction of the solutions is made possible by an ansatz which explores the conformal symmetry and a $U(1)$ subgroup of the area preserving diffeomorphism group.

Keywords: Integrable Field Theories, Integrable Hierarchies, Sigma Models.

Contents

1. Introduction

The development of exact methods is very important for the understanding of several non-perturbative aspects of field theories of physical interest. Soliton solutions are in one way or the other connected to the few advances in this area. Since the appearance of such special type of solutions require a large degree of symmetries, the integrability properties play a fundamental role. In low dimensions the structures responsible for the integrability and soliton solutions is reasonably well understood. In higher dimensions, although some exact results have been obtained, the essence of the structures involving solitons in such field theories is far from clear. For gauge theories the most important and relevant results were obtained using supersymmetry and duality concepts [1]. However, for general Lorentz invariant field theories in dimensions higher than two, mathematical concepts like loop spaces may play an important role as well, specially in the classical integrability structures [2]. The study of models that present exact soliton solutions is very important as a laboratory to test ideas and methods.

In this paper we consider a non-linear field theory on the space-time $S^3 \times \mathbb{R}$, with the fields taking values on the sphere S^2, and the Lagrangean being the pull-back of the area form on the target space. The theory is integrable in the sense of possessing a zero curvature representation within the approach of [2], an infinite number of conserved currents, and exact soliton solutions. From the topological point of view it belongs to a class of models, with target space S^2, presenting solitons with non-trivial Hopf topological charges, and where the best known example is the Skyrme-Faddeev model [3, 4].

We construct an infinite number of exact soliton solutions with finite energy, non-trivial Hopf topological charges, and that spin with a frequency ω that is bounded above by a quantity inversely proportional to the radius r_0 of S^3. Our work complements that of ref. [5] where time dependent solitons for the same model were also constructed. In addition, our solutions present properties similar to those of the solitons constructed in [6] for a theory with the same Lagrangean but on Minkowski space-time.

The action of the model we consider is given by

$$S = -\frac{1}{e^2} \int dt \int_{S^3} d\Sigma \; H_{\mu\nu}^2 \tag{1.1}$$

where $H_{\mu\nu}$ is the pull-back of the area form on S^2 given by

$$H_{\mu\nu} \equiv -2i \frac{(\partial_\mu u \partial_\nu u^* - \partial_\nu u \partial_\mu u^*)}{(1+ \mid u \mid^2)^2} = \vec{n} \cdot (\partial_\mu \vec{n} \wedge \partial_\nu \vec{n}) \tag{1.2}$$

where u is a complex scalar field, and $\vec{n}$ ($\vec{n}^2 = 1$) is a triplet of real scalar fields living on S^2. They are related by the stereographic projection

$$\vec{n} = \frac{1}{1+ \mid u \mid^2} \left(u + u^*, -i\,(u - u^*)\,, \mid u \mid^2 - 1 \right) . \tag{1.3}$$

The Euler-Lagrange equations associated to (1.1) are given by

$$\partial_\mu \mathcal{K}^\mu = 0 \tag{1.4}$$

and its complex conjugate, and where

$$\mathcal{K}_\mu = H_{\mu\nu} \partial^\nu u\,. \tag{1.5}$$

The action (1.1) and equations of motion (1.4) are invariant under the group $SO(4,2)$ of conformal transformations on $S^3 \times \mathbb{R}$. They are also invariant under the infinite group of area preserving diffeomorphisms of the target space S^2. The associated Noether currents are given by [7]

$$J_\mu^G = \frac{\delta G}{\delta u} \mathcal{K}_\mu + \frac{\delta G}{\delta u^*} \mathcal{K}_\mu^* \tag{1.6}$$

with G being any functional of u and u^*, but not of their derivatives. The conservation of the currents (1.6) is a consequence of the equations of motion (1.4), and the fact that

$$\mathcal{K}_\mu \partial^\mu u = 0 \qquad\qquad \mathcal{K}_\mu \partial^\mu u^* + \mathcal{K}_\mu^* \partial^\mu u = 0\,. \tag{1.7}$$

The model (1.1) has been considered in ref. [5] where an infinite set of soliton solutions, static and time dependent, with non-trivial Hopf charges have been constructed. The basic ingredient of the construction in [5] was the use of special coordinates on $S^3 \times \mathbb{R}$ which in its turn leads to a powerful ansatz. The group of area preserving diffeomorphisms has a $U(1)_\alpha$ subgroup generated by the phase transformations $u \to e^{i\alpha}\, u$, with α constant. Considering commuting $U(1)_{\varphi_i}$'s subgroups in the conformal group one build an ansatz where the field configurations are invariant under each of the diagonal subgroups of $U(1)_\alpha \otimes U(1)_{\varphi_i}$ [8].

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The special coordinates are then chosen in such a way that the transformations associated to the $U(1)_{\varphi_i}$'s correspond to translation along them. However, for the method to work, the subgroups $U(1)_{\varphi_i}$'s do not have to be compact as $U(1)_\alpha$ is, and so translations along non-compact directions, as time for instance, can be included into the ansatz. A more detailed account of these methods, valid on any space-time, will be given in ref. [9].

The paper is organized as follow: in section 2 we present the special coordinates and the corresponding ansatz, which reduces the four dimensional non-linear partial differential equations (1.4) into a single linear ordinary differential equation for a real profile function. The solutions with appropriated boundary conditions is then constructed. In section 3 we calculate the Hopf topological charges for those solutions. The Noether charges associated to the area preserving diffeomorphisms of S^2 are calculated in sec 4. The angular momenta associated to the $SO(4)$ rotations on S^3 are evaluated in section 5, and the energies of the solutions are given in section 6. We give an appendix with the details of the calculations involved in the evaluation of the Noether charges.

2. Soliton solutions

Following [5] we then introduce a set of coordinates in the space-time $S^3 \times \mathbb{R}$ such that the metric is given by

$$ds^2 = c^2\, dt^2 - r_0^2 \left(\frac{dz^2}{4\, z\,(1-z)} + (1-z)\, d\varphi_1{}^2 + z\, d\varphi_2{}^2 \right) \tag{2.1}$$

where t is the time, c is the speed of light, z and φ_i, $i = 1, 2$, are coordinates on the sphere S^3, and $0 \leq z \leq 1$, $0 \leq \varphi_i \leq 2\pi$, and r_0 is the radius of the sphere S^3. Embedding S^3 on $\mathbb{R}^4$ we get that the Cartesian coordinates of the points of S^3 are

$$\begin{aligned} x_1 &= r_0 \sqrt{z}\, \cos\varphi_2 \qquad & x_3 &= r_0 \sqrt{1-z}\, \cos\varphi_1 \\ x_2 &= r_0 \sqrt{z}\, \sin\varphi_2 \qquad & x_4 &= r_0 \sqrt{1-z}\, \sin\varphi_1 \,. \end{aligned} \tag{2.2}$$

The ansatz is given by [8, 5, 10, 9][1]

$$u = \sqrt{\frac{1-g}{g}}\, e^{i(m_1\varphi_1 + m_2\varphi_2 + \omega t)} \tag{2.3}$$

where m_i, $i = 1, 2$, are arbitrary integers, ω is a real frequency, and g is a real profile function. In addition, we require that $0 \leq g \leq 1$ for the factor $\sqrt{(1-g)/g}$ to be real. The form of that factor is important because it implies g satisfies a linear ordinary differential equation. Indeed, replacing (2.3) into (1.4) one gets

$$\partial_z \left(\Omega\, \partial_z\, g \right) = 0 \tag{2.4}$$

with

$$\Omega = m_1^2\, z + m_2^2\, (1-z) - \frac{r_0^2 \omega^2}{c^2}\, z\,(1-z) \,. \tag{2.5}$$

Since we have to satisfy the condition $0 \leq g \leq 1$, we can not have zeroes of Ω lying in the allowed range for z, i.e., $0 \leq z \leq 1$.

[1] On the completion of this paper we became aware of ref. [11] where that ansatz, with a different profile function, was used on some other models on $S^3 \times \mathbb{R}$.

For $\omega = 0$ one has that the zeroes of Ω are $z = 0$ for $m_2 = 0$, $z = 1$ for $m_1 = 0$, and they lie in the intervals $z > 1$ or $z < 0$, for non-vanishing m_1 and m_2. Therefore, we have to discard the cases $m_1 = 0$ or $m_2 = 0$. The solutions satisfying the boundary conditions $g\,(0) = 0$ and $g\,(1) = 1$, for $\omega = 0$ are then

$$g = z\;; \qquad\qquad \text{for} \qquad \omega = 0 \quad m_1^2 = m_2^2 \tag{2.6}$$

and

$$g = \frac{\ln\left(\left(q^2 - 1\right)\, z + 1\right)}{\ln q^2}\;; \qquad\qquad \text{for} \qquad \omega = 0 \quad m_1^2 \neq m_2^2 \tag{2.7}$$

where

$$q \equiv \frac{\mid m_1 \mid}{\mid m_2 \mid} \tag{2.8}$$

These are the static solutions of [5].

For $\omega \neq 0$ we write $\Omega = \frac{r_0^2\omega^2}{c^2}\,(z - z_+)\,(z - z_-)$, with $z_\pm = -b \pm \sqrt{\Delta}$, and where

$$b = \frac{1}{2}\,(p_+ \, p_- - 1)\;; \qquad \Delta = \frac{1}{4}\,\left(p_+^2 - 1\right)\left(p_-^2 - 1\right)\;; \qquad p_\pm \equiv \frac{c}{r_0\,\omega}\,(m_1 \pm m_2) \tag{2.9}$$

Notice that b and Δ are invariant under the independent change of signs of m_1, m_2 and ω. The solutions for (2.4) are $g \sim \frac{1}{z - z_+} + \text{const.}$, for $z_+ = z_-$, and $g \sim \ln \frac{z - z_+}{z - z_-} + \text{const.}$ for $z_+ \neq z_-$. Therefore, we can not have one or both of the zeroes $z_\pm$ of Ω, lying in the interval $[0, 1]$, since g will have divergencies and so the condition $0 \leq g \leq 1$ will not be satisfied. A careful analysis shows that $z_\pm \in [0, 1]$, when both $p_\pm$ lie in the interval $[-1, 1]$, and so when $\frac{r_0^2\,\omega^2}{c^2} \geq (\mid m_1 \mid + \mid m_2 \mid)^2$. In addition, if $p_\pm^2 > 1$, then $z_+ = 0$ for $p_+ = p_-$, and $z_- = 1$ for $p_+ = -p_-$. Therefore, we can not have $m_1 = 0$ or $m_2 = 0$. That is also true in the case $\omega = 0$, and consequently, as we show below, it implies that all our solutions will have non-trivial Hopf topological charges.

Therefore the solutions for $\omega \neq 0$, satisfying the boundary conditions $g\,(0) = 0$ and $g\,(1) = 1$, are

1. In the cases where $p_\pm^2 > 1$ ($\Delta > 0$) one has

$$g = \left[\ln \frac{z + b - \sqrt{\Delta}}{z + b + \sqrt{\Delta}} - \ln \frac{b - \sqrt{\Delta}}{b + \sqrt{\Delta}}\right] / \ln \frac{a + \sqrt{\Delta}}{a - \sqrt{\Delta}}\;; \qquad \frac{r_0^2\,\omega^2}{c^2} < (\mid m_1 \mid - \mid m_2 \mid)^2 \tag{2.10}$$

   with $a = \frac{1}{2}\left(\frac{p_+^2 + p_-^2}{2} - 1\right)$.

2. In the cases where $p_+^2 = 1$ and $p_-^2 > 1$, or $p_-^2 = 1$ and $p_+^2 > 1$, ($\Delta = 0$) one has

$$g = \frac{q}{q - 1}\left(1 - \frac{1}{(q - 1)\; z + 1}\right)\;; \qquad \frac{r_0^2\,\omega^2}{c^2} = (\mid m_1 \mid - \mid m_2 \mid)^2 \tag{2.11}$$

   with q given in (2.8).

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3. In the cases where $p_+^2 < 1$ and $p_-^2 > 1$, or $p_+^2 > 1$ and $p_-^2 < 1$, $(\Delta < 0)$ one has

$$g = \left[\text{ArcTan}\frac{\sqrt{-\Delta}}{b} - \text{ArcTan}\frac{\sqrt{-\Delta}}{b+z}\right] / \text{ArcTan}\frac{\sqrt{-\Delta}}{a} \tag{2.12}$$

$$(\mid m_1 \mid - \mid m_2 \mid)^2 < \frac{r_0^2\,\omega^2}{c^2} < (\mid m_1 \mid + \mid m_2 \mid)^2$$

and where the ArcTan takes values between 0 and π.

In all those cases g is a monotonically increasing function of z starting with $g = 0$ at $z = 0$ and finishing with $g = 1$ at $z = 1$. Notice that given the values of m_1 and m_2, the type of profile function g is determined by the dimensionless quantity $\frac{r_0\,\omega}{c}$. In addition, the range in which such dimensionless quantity can vary is restricted by those same integers. Therefore, the frequency of rotation of the solution is inversely proportional to the radius r_0 of the sphere S^3. We point out that the period, $\frac{2\pi}{\omega}$, of such rotations is never shorter than the time that a light ray takes to travel along a maximum circle on S^3.

In all the cases in (2.10)–(2.12), we have that

$$\partial_z\, g = \frac{\beta}{\Omega} \tag{2.13}$$

with

$$\beta \equiv \left(m_1^2 + m_2^2 - \frac{r_0^2\,\omega^2}{c^2}\right)\frac{w}{\ln\frac{1+w}{1-w}}\;; \qquad\qquad w \equiv \frac{\sqrt{\Delta}}{a} \tag{2.14}$$

In the case where $\Delta = 0$, one has that β simplifies to $\beta = \mid m_1\, m_2 \mid$.

The form of the solution can be visualized by the surfaces in S^3 of constant n_3, the third component of $\vec{n} \in S^2$, given in (1.3). Notice that n_3 depends on $\mid u \mid^2$ which in its turn depends on g (see (2.3)). But since g is a monotonic function of z, it implies that constant n_3 means constant z. Therefore, the surfaces of constant n_3 can be obtained from (2.2) by fixing z and varying φ_1 and φ_2. Such surfaces are the same for all solutions. What changes from one solution to the other is the correspondence between z and n_3, determined through (2.10)–(2.12) by the triple (m_1, m_2, ω). Notice that such surfaces do not evolve in time, since $\mid u \mid^2$ is time independent. For all solutions, the pre-image of the north pole of S^2, $\vec{n} = (0, 0, 1)$ (and so $z = 0$), corresponds to a circle of radius r_0 on the plane x^3x^4, and the pre-image of the south pole of S^2, $\vec{n} = (0, 0, -1)$ (and so $z = 1$), corresponds to a circle of radius r_0 on the plane x^1x^2. The pre-image of a given $\vec{n}$ with $-1 < n_3 < 1$, is a two dimensional surface of a torus like shape.

3. The Hopf topological charge

At any fixed time t our solution defines a map from the physical space S^3 to the target space S^2, and so it is a Hopf map. The Hopf invariant, or linking number, is calculated as follow. First we define a map from the physical S^3 to another 3-sphere S_Z^3, as

$$Z = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1-g}\, e^{i\,(m_1\varphi_1 + \omega\, t)} \\ \sqrt{g}\, e^{-i\, m_2 \varphi_2} \end{pmatrix} \tag{3.1}$$

where w_1 and w_2 are two complex coordinates parametrizing S^3_Z, such that $Z^\dagger\, Z = \mid w_1 \mid^2 + \mid w_2 \mid^2 = 1$. Then we map S^3_Z into the target space S^2 as $u = w_1/w_2$. The Hopf invariant is defined through the integral

$$Q_H = \frac{1}{4\pi^2}\int_{S^3} d\Sigma\, \vec{A}\cdot\left(\vec{\nabla}\wedge\vec{A}\right) \tag{3.2}$$

where $\vec{\nabla}$ is gradient on the physical S^3,

$$d\Sigma = \frac{r_0^3}{2}\, dz\, d\varphi_1\, d\varphi_2 \tag{3.3}$$

is the volume element on S^3, and

$$\vec{A} = \frac{i}{2}\left(Z^\dagger\,\vec{\nabla}Z - \vec{\nabla}Z^\dagger\, Z\right) \tag{3.4}$$

Evaluating it one gets [5]

$$\vec{A} = -\frac{m_1}{r_0}\frac{(1-g)}{\sqrt{1-z}}\hat{\mathbf{e}}_{\varphi_1} + \frac{m_2}{r_0}\frac{g}{\sqrt{z}}\hat{\mathbf{e}}_{\varphi_2}\,; \qquad \vec{\nabla}\wedge\vec{A} = \frac{2}{r_0^2}\partial_z\, g\left(-m_2\sqrt{1-z}\hat{\mathbf{e}}_{\varphi_1} + m_1\sqrt{z}\hat{\mathbf{e}}_{\varphi_2}\right) \tag{3.5}$$

and so, $\vec{A}$ is time independent. Therefore

$$Q_H = m_1\, m_2\,(g\,(1) - g\,(0)) = m_1\, m_2\,. \tag{3.6}$$

That is valid for all solutions (2.3) and (2.10)–(2.12), since they all satisfy the boundary condition $g\,(1) = 1$ and $g\,(0) = 0$. Since, from our considerations in section 2, the integers m_1 and m_2 can never vanish, all solutions have non-trivial Hopf topological charges.

4. The Noether charges

One can check that the time component of the vector $\mathcal{K}_\mu$, defined in (1.5), evaluated on the ansatz configurations (2.3) is given by

$$\mathcal{K}_0 = -\frac{4}{r_0^2}\,\frac{\omega}{c}\,\frac{z\,(1-z)}{g\,(1-g)}\,(\partial_z\, g)^2\;\; u\,. \tag{4.1}$$

Therefore, the density of the Noether charges associated to (1.6), with a functional of the form $G = u^m\, u^{*n}$ is given by (m and n are integers for G to be single valued for any configuration (2.3))

$$J_0^{(m,n)} = -\,(m+n)\,\frac{4}{r_0^2}\,\frac{\omega}{c}\,\frac{z\,(1-z)}{g\,(1-g)}\,(\partial_z\, g)^2\,\left(\frac{1-g}{g}\right)^{\frac{(m+n)}{2}}\; e^{i(m-n)(m_1\varphi_1+m_2\varphi_2+\omega t)}\,. \tag{4.2}$$

So, if $m \neq n$, the corresponding charge will vanish since the integral of the phase factor with the volume element (3.3) is zero. Consequently, the relevant Noether charges are those where G is a functional of the norm of u, or equivalently a functional of g. Such infinite set of charges have vanishing Poisson brackets [7], and so should be important for the integrability of the model. In fact, such abelian subalgebra was shown to be connected to constraints leading to integrable submodels of some theories with target space S^2 [12]. Using (1.6) and (4.1) one then gets that, for $G \equiv G(g)$,

$$Q^G = \int_{S^3} d\Sigma\, J_0^G = 16\,\pi^2\,\frac{r_0\,\omega}{c}\int_0^1 dz\, z\,(1-z)\,(\partial_z\, g)^2\,\frac{\delta G}{\delta g}\,. \tag{4.3}$$

Choosing $G = g^n/16\pi^2 n!$, with n a positive integer, we get that

$$Q^{(n)} = \frac{r_0\,\omega}{c}\,F^{(n)}(w) \qquad\qquad n = 1,2,3\ldots \tag{4.4}$$

where

$$F^{(n)}(w) \equiv \frac{1}{\left(\ln\frac{1+w}{1-w}\right)^{n+1}}\left[-2\,\epsilon_-(n) + \sum_{l=1}^{n}\left(\frac{\epsilon_+(n-l)}{w} - \epsilon_-(n-l)\right)\frac{1}{l!}\left(\ln\frac{1+w}{1-w}\right)^l\right] \tag{4.5}$$

with $\epsilon_\pm(n) \equiv (1 \pm (-1)^n)/2$, and w defined in (2.14). In appendix A we give the details of the calculations leading to (4.4). In the case where $\Delta = 0$, we have that the Noether charges simplify to $Q^{(n)} = \frac{r_0\,\omega}{c}\,\frac{n}{(n+2)!}$.

5. The angular momentum

The action (1.1) is invariant under the conformal group $SO(4,2)$ of the space-time $S^3 \times \mathbb{R}$. Such group contains the subgroup $SO(4)$ of rotations on the sphere S^3. In ref. [9] we shall present a more detailed study of how the $SO(4,2)$ conformal transformations, and $SO(4)$ rotations, are realized in terms of the coordinates (2.1)–(2.2). Here we present the six Noether charges associated to the $SO(4)$ symmetry. According to (2.2), the rotations on the planes x^1x^2 and x^3x^4 correspond to the translations on the angles φ_2 and φ_1 respectively. The density of the corresponding Noether charges, for the ansatz configurations (2.3), are

$$J^0_{\varphi_i} = \frac{64\,\omega}{e^2\,r_0^2\,c}\,m_i\,z\,(1-z)\,(\partial_z\,g)^2 \qquad i = 1,2\,. \tag{5.1}$$

Integrating it with the measure (3.3), for the solutions (2.10)–(2.12), one gets the charges

$$Q_{\varphi_i} = \frac{128\,\pi^2}{e^2}\,\frac{r_0\,\omega}{c}\,m_i\,F^{(1)}(w) \qquad i = 1,2 \tag{5.2}$$

with $F^{(1)}(w)$ as in (4.5), i.e.

$$F^{(1)}(w) = \frac{1}{w\,\ln\frac{1+w}{1-w}} - \frac{2}{\left(\ln\frac{1+w}{1-w}\right)^2}\,. \tag{5.3}$$

The density of the Noether charges, for the ansatz configurations (2.3), associated to the rotations on the remaining four planes x^ix^j, are given by

$$J^0_{\delta_1\delta_2} = \frac{64\,\omega}{e^2\,r_0^2\,c}\,z\,(1-z)\,(\partial_z\,g)^2\left[-m_1\sqrt{\frac{z}{1-z}}\,\sin(\varphi_1+\delta_1)\,\cos(\varphi_2+\delta_2)\right.$$
$$\left. +\; m_2\sqrt{\frac{1-z}{z}}\,\cos(\varphi_1+\delta_1)\sin(\varphi_2+\delta_2)\right] \tag{5.4}$$

with $\delta_i = 0, \frac{\pi}{2}$, $i = 1,2$. Therefore, the corresponding Noether charges vanish since the densities are periodic in the angles φ_i.

6. The energy

The Hamiltonian density associated to (1.1), evaluated on the ansatz configurations (2.3), is given by

$$\mathcal{H} = \frac{32}{e^2\, r_0^4}\, (\partial_z\, g)^2 \left[m_1^2\, z + m_2^2\, (1-z) + \frac{r_0^2\, \omega^2}{c^2}\, z\, (1-z) \right] . \tag{6.1}$$

For the static solutions (2.6)–(2.7) the energy is given by

$$E = \int_{S^3} d\Sigma\, \mathcal{H} = \frac{64\, \pi^2}{e^2\, r_0} \mid m_1 m_2 \mid \frac{(q-1/q)}{\ln q^2} \tag{6.2}$$

which is the result of [5]. In the limit $q \to 1$ ($m_1 \to \pm m_2$), one has $E \to \frac{64\,\pi^2}{e^2\, r_0}\, m_1^2$, and so the energy becomes proportional to the modulus of Hopf charge (3.6).

Using (2.5), (2.13) and (3.3) one gets that the energy, for the ansatz configurations (2.3), can be written as

$$E = \frac{64\, \pi^2}{e^2\, r_0} \int_0^1 dz \left[\beta\, \partial_z\, g + 2\, \frac{r_0^2\, \omega^2}{c^2}\, z\, (1-z)\, (\partial_z\, g)^2 \right] . \tag{6.3}$$

Therefore, for the solutions (2.10)–(2.12), it becomes

$$E = \frac{64\, \pi^2}{e^2\, r_0} \left[\beta + 2\, \frac{r_0^2\, \omega^2}{c^2}\, F^{(1)}\, (w) \right] \tag{6.4}$$

with $F^{(1)}\,(w)$ given in (5.3), and β in (2.14).

In the case where $\Delta = 0$, or $\frac{r_0^2\,\omega^2}{c^2} = (\mid m_1 \mid - \mid m_2 \mid)^2$, one has $\beta = \mid m_1\, m_2 \mid$, and $F^{(1)}\,(0) = 1/6$. Then the energy simplifies to

$$E = \frac{64\, \pi^2}{3\, e^2\, r_0} \left(m_1^2 + m_2^2 + \mid m_1\, m_2 \mid \right) . \tag{6.5}$$

The energy (6.4) is invariant under the independent change of the signs of m_1, m_2 and ω. In addition, it is invariant under the interchange $m_1 \leftrightarrow m_2$. Consequently, the energy is 16-fold degenerate for $m_1 \neq m_2$ and $\omega \neq 0$, 8-fold degenerate for $m_1 \neq m_2$ and $\omega = 0$, 4-fold degenerate for $m_1 = m_2$ and $\omega \neq 0$, and 2-fold degenerate for $m_1 = m_2$ and $\omega = 0$. As we noticed above, there are no physically acceptable solutions for $m_1 = 0$ or $m_1 = 0$. All the degenerate solutions can be completely distinguished from one another by the values of the topological Hopf charge (3.6), Noether charges (4.4), and angular momenta (5.2).

In figure 1 we give the plot of the energy as a function of the dimensionless quantity $r_0^2\, \omega^2/c^2$ for some values of (m_1, m_2). Notice that for every solution labelled by the triple (m_1, m_2, ω), the frequency ω, as discussed in (2.10)–(2.12), can vary as $-\,(\mid m_1 \mid + \mid m_2 \mid) < \frac{r_0\,\omega}{c} < \mid m_1 \mid + \mid m_2 \mid$. The spectrum of energy of our solutions have three main features: *i)* For $\omega \to 0$ the energy (6.4) approaches the static energy (6.2), *ii)* for fixed m_1 and m_2 the energy grows monotonically with ω^2, and *iii)* the energy diverges faster, as a function of ω, for solutions with lower $\mid m_1 \mid + \mid m_2 \mid$.

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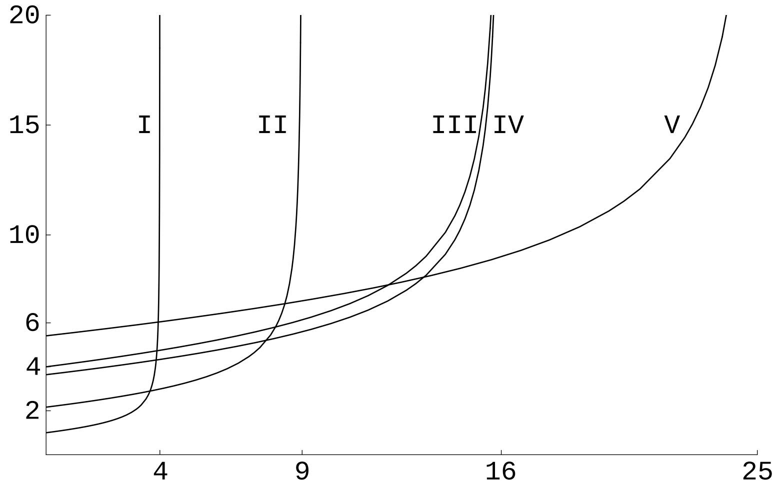

Figure 1: Plot of the energy (6.4) (in units of $64\,\pi^2/e^2\,r_0$) as a function of $r_0^2\,\omega^2/c^2$ for some values of (m_1, m_2). The curve I corresponds to $(m_1, m_2) = (1, 1)$, curve II to $(m_1, m_2) = (1, 2)$, curve III to $(m_1, m_2) = (1, 3)$, curve IV to $(m_1, m_2) = (2, 2)$, and curve V to $(m_1, m_2) = (1, 4)$.

A. The Noether charge calculations

In order to obtain the Noether charges (4.3) we have to evaluate the integral

$$I^G = \int_0^1 dz\, z\,(1-z)\,\left(\partial_z\, g\right)^2\,\frac{\delta G}{\delta g}\,. \tag{A.1}$$

It is convenient to change the integration variable as

$$y = \frac{1-z}{z} \tag{A.2}$$

and so

$$I^G = \int_0^\infty dy\, y\,\left(\partial_y\, g\right)^2\,\frac{\delta G}{\delta g}\,. \tag{A.3}$$

Notice that the integrand of I^G has its form unchanged under the transformation $y \rightarrow 1/y$, or equivalently $z \rightarrow 1-z$. That is important in relating the Noether charges for solutions with interchanged boundary conditions at $z = 0$ and $z = 1$. In particular, the integral (A.3) is the one appearing in the expression for the Noether charges of the model considered in [6]. The solutions (2.10)–(2.12), in terms of y, are written as

$$g = \frac{1}{\ln \frac{y_-}{y_+}}\,\ln \frac{y-y_-}{y-y_+}\,; \qquad\qquad y_\pm = \frac{-a \pm \sqrt{\Delta}}{a-b} \tag{A.4}$$

and

$$\partial_y\, g = -\frac{\beta}{\Lambda} \tag{A.5}$$

where β is given in (2.14), and

$$\Lambda \equiv m_1^2\,(1+y) + m_2^2\,y\,(1+y) - \frac{r_0^2\,\omega^2}{c^2}\,y = m_2^2\,(y-y_+)\,(y-y_-)\,. \tag{A.6}$$

Therefore, for $G = g^n/16\pi^2 n!$, one has that (A.3) becomes

$$I^G = \frac{\beta^2}{16\pi^2\,(n-1)!m_2^4}\,\frac{1}{\left(\ln\frac{y_-}{y_+}\right)^{n-1}}\int_0^\infty dy\,\frac{y}{\left(y-y_+\right)^2\left(y-y_-\right)^2}\,\left(\ln\frac{y-y_-}{y-y_+}\right)^{n-1}\,. \tag{A.7}$$

One can check that

$$\frac{d\,k\,(n)}{dy} = \frac{y}{\left(y-y_+\right)^2\left(y-y_-\right)^2}\,\left(\ln\frac{y-y_-}{y-y_+}\right)^{n-1} \tag{A.8}$$

with

$$\begin{aligned}k\,(n) \equiv -\frac{(n-1)!}{\left(y_+-y_-\right)^2}\quad&\left[\frac{y_-}{y-y_-} - (-1)^n\,\frac{y_+}{y-y_+} - \frac{y_++y_-}{y_+-y_-}\,\frac{1}{n!}\,\left(\ln\frac{y-y_-}{y-y_+}\right)^n\right.\\ &\left.+\sum_{l=1}^{n-1}\frac{1}{l!}\,\left(\ln\frac{y-y_-}{y-y_+}\right)^l\,P\,(n-1-l)\right]\end{aligned} \tag{A.9}$$

with

$$\begin{aligned}P\,(n) &= \frac{y^2-y_+\,y_-}{\left(y-y_+\right)\left(y-y_-\right)} &&\text{for } n \text{ even}\\ P\,(n) &= -\frac{\left(y_++y_-\right)\left(y^2+y_+\,y_-\right)-4\,y\,y_+\,y_-}{\left(y-y_+\right)\left(y-y_-\right)\left(y_+-y_-\right)} &&\text{for } n \text{ odd}\,.\end{aligned} \tag{A.10}$$

Therefore, we have that

$$I^G = \frac{\beta^2}{16\pi^2\,(n-1)!m_2^4}\,\frac{1}{\left(\ln\frac{y_-}{y_+}\right)^{n-1}}\,\left[k\,(n)\,|_{y=\infty}\,-k\,(n)\,|_{y=0}\right] = \frac{1}{16\pi^2}\,F^{(n)}\,(w) \tag{A.11}$$

with $F^{(n)}\,(w)$ given in (4.5).

Acknowledgments

A.C.R.B. is supported by a CNPq scholarship, and L.A.F. is partially supported by a CNPq grant.

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